AUSTRALIAN CURRICULUM, ASSESSMENT AND REPORTING AUTHORITY

Senior Secondary Australian Curriculum

Specialist Mathematics Glossary

Unit 1

Combinatorics

Arranging *n* objects in an ordered list
The number of ways to arrange *n* different objects in an ordered list is given by
$n(n-1)(n-2) \times \ldots \times 3 \times 2 \times 1 = n!$

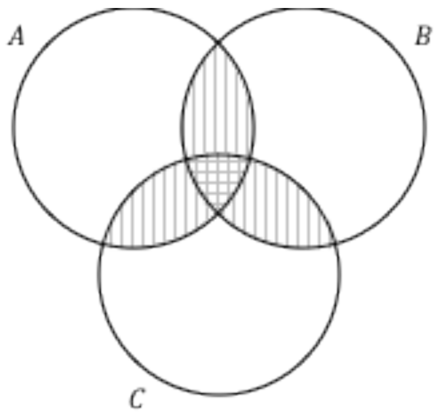
Combinations (Selections)
The number of selections of *n* objects taken *r* at a time (that is, the number of ways of selecting *r* objects out of *n*) is denoted by ${}^nC_r = \binom{n}{r}$ and is equal to $\frac{n!}{r!(n-r)!}$

Inclusion – exclusion principle

- Suppose *A* and *B* are subsets of a finite set *X* then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Suppose *A*, *B* and *C* are subsets of a finite set *X* then



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

This result can be generalised to 4 or more sets.

Multiplication principle
Suppose a choice is to be made in two stages. If there are *a* choices for the first stage and *b* choices for the second stage, no matter what choice has been made at the first stage, then there are *ab* choices altogether. If the choice is to be made in *n* stages and if for each *i,* there are a_i choices for the i^{th} stage then there are $a_1a_2\ldots a_n$ choices altogether.

Pascal's triangle
Pascal's triangle is an arrangement of numbers. In general the n^{th} row consists of the binomial coefficients ${}^nC_r$ or $\binom{n}{r}$ with the $r = 0, 1, \ldots, n$

$$
\begin{array}{ccccccccccc}
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

In Pascal's triangle any term is the sum of the two terms 'above' it.
For example 10 = 4 + 6.
Identities include:

- The recurrence relation, ${}^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$
- ${}^nC_k = \frac{n}{k}\,{}^{n-1}C_{k-1}$

Permutations
A permutation of n objects is an arrangement or rearrangement of n objects (order is important). The number of arrangements of n objects is $n!$ The number of permutations of n objects taken r at a time is denoted ${}^nP_r$ and is equal to

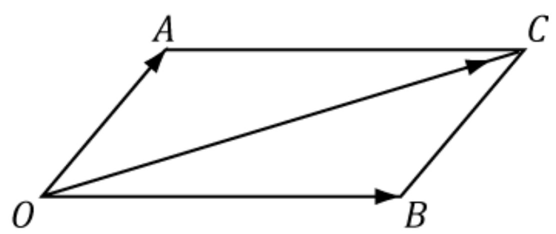
$n(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!}$.

Pigeon-hole principle
If there are n pigeon holes and $n + 1$ pigeons to go into them, then at least one pigeon hole must get 2 or more pigeons.

Vectors in the plane

Addition of vectors (see Vector for definition and notation)

Given vectors **a** and **b** let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be directed line segments that represent **a** and **b.** They have the same initial point O. The sum of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is the directed line segment $\overrightarrow{OC}$ where C is a point such that $OACB$ is a parallelogram. This is known as the **parallelogram rule**.



If $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ then $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2)$
In component form if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ then
$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j}$

Properties of vector addition:

- **a** + **b** = **b** + **a** (commutative law)
- (**a** + **b**) + **c** = **a** + (**b** + **c**) (associative law)
- **a** + **0** = **0** + **a** = **a**
- **a** + (–**a**) = **0**

Magnitude of a vector (see Vector for definition and notation)
The magnitude of a vector **a** is the length of any directed line segment that represents **a.** It is denoted by |**a**|.

Multiplication by a scalar
Let **a** be a non-zero vector and *k* a positive real number (scalar) then the scalar multiple of **a** by *k* is the vector *k**a*** which has magnitude |*k* ||**a**| and the same direction as **a.** If *k* is a negative real number, then *k* **a** has magnitude | *k* | |**a**| and but is directed in the opposite direction to **a**. (see **negative of a vector**)
Some properties of scalar multiplication are:

- *k* (**a** + **b**) = *k* **a** + *k* **b**
- *h*(*k* (**a**)) = (*hk*)**a**
- 1**a** = **a**

Negative of a vector (see Vector for definition and notation)

Given a vector **a**, let $\overrightarrow{AB}$ be a directed line segment representing **a.** The negative of **a**, denoted by –**a,** is the vector represented by $\overrightarrow{BA}$. The following are properties of vectors involving negatives:

- **a** + (–**a**) = (–**a**) + **a** = **0**
- –(–**a**) = **a**

Scalar product (see Vector for definition and notation)
a = (a_1, a_2) and **b** = (b_1, b_2) then the scalar product **a**.**b** is the real number
$a_1 b_1 + a_2 b_2$. The geometrical interpretation of this number is **a.b** =|**a**||**b**|cos (θ) where θ is the angle 'between' **a** and **b**
When expressed in **i**, **j**, notation, if **a** = a_1**i** + a_2**j** and **b** = b_1**i** + b_2**j** then
a.b = $a_1 b_1 + a_2 b_2$
Note |**a**| = $\sqrt{\mathbf{a}.\mathbf{a}}$

Subtraction of vectors (see Vector for definition and notation)

a – b = a + (–b)

Unit vector (see Vector for definition and notation)
A unit vector is a vector with magnitude 1. Given a vector **a** the unit vector in the same direction as **a** is $\frac{1}{|\mathbf{a}|}$**a.** This vector is often denoted as $\hat{\mathbf{a}}$.

Vector projection (see Vector for definition and notation)
Let **a** and **b** be two vectors and write θ for the angle between them. The projection of a vector **a** on a vector **b** is the vector

$|\mathbf{a}| \cos\theta\, \hat{\mathbf{b}}$ where $\hat{\mathbf{b}}$ is the unit vector in the direction of **b.**

The projection of a vector **a** on a vector **b** is $(\mathbf{a}.\hat{\mathbf{b}})\,\hat{\mathbf{b}}$ where $\hat{\mathbf{b}}$ is the unit vector in the direction of **b.**

This projection is also given by the formula $\frac{\mathbf{a}.\mathbf{b}}{\mathbf{b}.\mathbf{b}}\mathbf{b}$.

Vector

In Physics the name vector is used to describe a physical quantity like velocity or force that has a magnitude and direction.

A vector is an entity **a** which has a given length (magnitude) and a given direction. If $\overrightarrow{AB}$ is a directed line segment with this length and direction, then we say that $\overrightarrow{AB}$ represents **a.**

If $\overrightarrow{AB}$ and $\overrightarrow{CD}$ represent the same vector, they are parallel and have the same length.
The **zero vector** is the vector with length zero.
In two dimensions, every vector can be represented by a directed line segment which begins at the origin. For example, the vector $\overrightarrow{BC}$ from B(1,2) to C(5,7) can be represented by the directed line segment $\overrightarrow{OA}$, where A is the point (4,5). The **ordered pair** notation for a vector uses the co-ordinates of the end point of this directed line segment beginning at the origin to denote the vector, so $\overrightarrow{BC}$ = (4,5) in ordered pair notation. The same vector can be represented in **column vector** notation as $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

Geometry

Glossary for Proof

Contradiction-Proof by
Assume the opposite (**negation**) of what you are trying to prove. Then proceed through a logical chain of argument till you reach a demonstrably false conclusion. Since all the reasoning is correct and a false conclusion has been reached the only thing that could be wrong is the initial assumption. Therefore the original statement is true.
For example: the result $\sqrt{2}$ is irrational can be proved in this way by first assuming $\sqrt{2}$ is rational.
The following are examples of results that are often proved by contradiction:

- If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.
- If an interval (line segment) subtends equal angles at two points on the same side of the interval (line segment), then the two points and the endpoints of the interval are concyclic.

Implication and Converse

Implication: if P then Q Symbol: $P \Rightarrow Q$
Examples:

- If a quadrilateral is a rectangle then the diagonals are of equal length and they bisect each other.
- If $x = 2$ then $x^2 = 4$.
- If an animal is a kangaroo then it is a marsupial.
- If a quadrilateral is cyclic then the opposite angles are supplementary.

Converse of a statement The converse of the statement 'If P then Q' is 'If Q then P' Symbolically the converse of $P \Rightarrow Q$ is: $Q \Rightarrow P$ or $P \Leftarrow Q$
The converse of a true statement need not be true.
Examples:

- **Statement:** If a quadrilateral is a rectangle then the diagonals are of equal length and they bisect each other.

  Converse statement: If the diagonals of a quadrilateral are of equal length and bisect each other then the quadrilateral is a rectangle. (In this case the converse is true.)

- **Statement:** If $x = 2$ then $x^2 = 4$.

  Converse statement: If $x^2 = 4$ then $x = 2$. (In this case the converse is false.)

- **Statement:** If an animal is a kangaroo then it is a marsupial.

  Converse statement: If an animal is a marsupial then it is a kangaroo. (In this case the converse is false.)

- **Statement:** If a quadrilateral is cyclic then the opposite angles are supplementary.

  Converse statement: If the opposite angles of a quadrilateral are supplementary then the quadrilateral is cyclic. (In this case the converse is true.)

Contrapositive

The contrapositive of the statement 'If P then Q' is 'If not Q then not P'. The contrapositive of a true statement is also true. (not Q is the **negation** of the statement Q)

Examples:

- **Statement:** A rectangle is a quadrilateral that has diagonals of equal length and the diagonals bisect each other.

  Contrapositive: If the diagonals of a quadrilateral are not of equal length or do not bisect each other then the quadrilateral is not a rectangle.

- **Statement:** If $x = 2$ then $x^2 = 4$.

  Contrapositive: If $x^2 \neq 4$ then $x \neq 2$.

- **Statement:** A kangaroo is a marsupial.

Contrapositive: If an animal is not a marsupial then it is not a kangaroo.

- **Statement:** The opposite angles of a cyclic quadrilateral are supplementary

  Contrapositive: If the opposite angles of quadrilateral are not supplementary then the quadrilateral is not cyclic.

Counterexample

A Counterexample is an example that demonstrates that a statement is not true.
Examples:

- **Statement**: If $x^2 = 4$ then $x = 2$.

  Counterexample: $x = -2$ provides a counterexample.

- **Statement**: If the diagonals of a quadrilateral intersect at right angles then the quadrilateral is a rhombus.

  Counterexample: A kite with the diagonals not bisecting each other is not a rhombus. Such a kite provides a counterexample to the statement. The diagonals of a kite do intersect at right angles.

- **Statement**: Every convex quadrilateral is a cyclic quadrilateral.

  Counterexample: A parallelogram that is not a rectangle is convex, but not cyclic.

Equivalent statements
Statements P and Q are equivalent if $P \Rightarrow Q$ and $Q \Rightarrow P$. The symbol $\Leftrightarrow$ is used. It is also written as P if and only if Q or P iff Q.
Examples:

- A quadrilateral is a rectangle if and only if the diagonals of the quadrilateral are of equal length and bisect each other.

- A quadrilateral is cyclic if and only if opposite angles are supplementary.

Negation
If P is a statement then the statement 'not P', denoted by $\neg P$ is the negation of P. If P is the statement 'It is snowing.' then $\neg P$ is the statement 'It is not snowing.'.

Quantifiers

For all (For each)
Symbol $\forall$

- For all real numbers x, $x^2 \geq 0$. ($\forall$ real numbers x, $x^2 \geq 0$.)

- For all triangles the sum of the interior angles is 180° .($\forall$ triangles the sum of the interior angles is 180°.)

- For each diameter of a given circle each angle subtended at the circumference by that diameter is a right angle.

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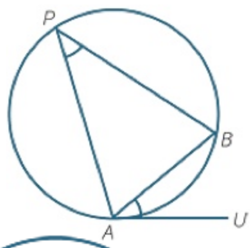
There exists
Symbol ∃

- There exists a real number that is not positive (∃ a real number that is not positive.)
- There exists a prime number that is not odd. (∃ a prime number that is not odd.)
- There exists a natural number that is less than 6 and greater than 3.
- There exists an isosceles triangle that is not equilateral.

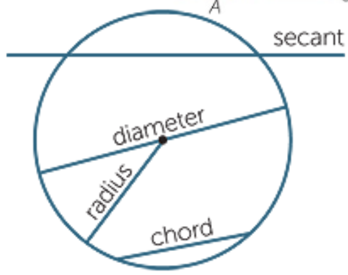
The quantifiers can be used together.
For example: $\forall\ x \geq 0, \exists\ y \geq 0$ such that $y^2 = x$.

Glossary of Geometric Terms and Listing of Important Theorems

Alternate segment
The word 'alternate' means 'other'. The chord *AB* divides the circle into two segments and *AU* is tangent to the circle. Angle *APB* 'lies in' the segment on the other side of chord *AB* from angle *BAU.* We say that it is in the **alternate segment**.



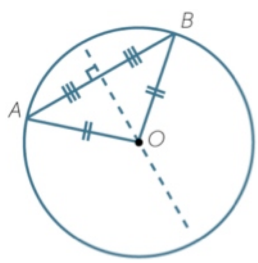
Cyclic quadrilateral
A **cyclic quadrilateral** is a quadrilateral whose vertices all lie on a circle.
Lines and line segments associated with circles
Any line segment joining a point on the circle to the centre is called a **radius**. By the definition of a circle, any two radii have the same length called the radius of the circle. Notice that the word 'radius' is used to refer both to these intervals and to the common length of these intervals.



- An interval joining two points on the circle is called a **chord**.
- A chord that passes through the centre is called a **diameter**. Since a diameter consists of two radii joined at their endpoints, every diameter has length equal to twice the radius. The word 'diameter' is use to refer both to these intervals and to their common length.

A line that cuts a circle at two distinct points is called a **secan**t. Thus a chord is the interval that the circle cuts off a secant, and a diameter is the interval cut off by a secant passing through the centre of a circle.

Circle Theorems
Result 1
Let *AB* be a chord of a circle with centre *O*.
The following three lines coincide:

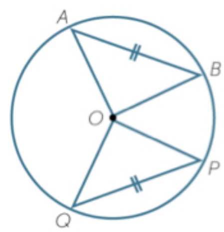


- The bisector of the angle ∠*AOB* subtended at the centre by the chord.
- The line segment (interval) joining *O* and the midpoint of the chord *AB*.
- The perpendicular bisector of the chord *AB*.

Result 2

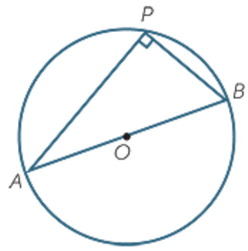
- Equal chords of a circle subtend equal angles at the centre.

  In the diagram shown ∠*AOB* = ∠*POQ*.



Result 3

- An angle in a semicircle is a right angle.



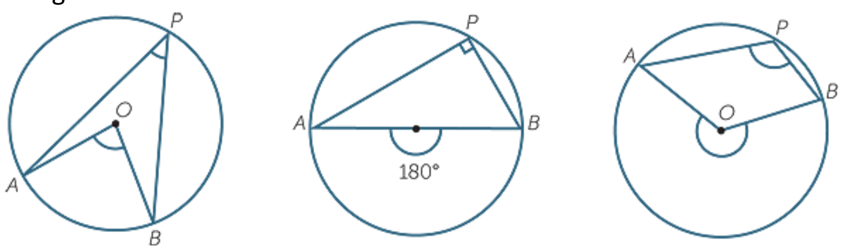
Let *AOB* be a diameter of a circle with centre *O*, and let *P* be any other point on the circle. The angle ∠*APB* subtended at *P* by the diameter *AB* is called an **angle in a semicircle.**

Converse

- The circle whose diameter is the hypotenuse of a right-angled triangle passes through all three vertices of the triangle.

Result 4

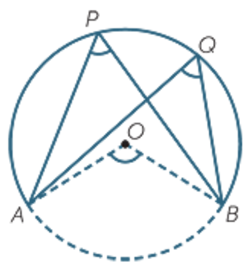
- An angle at the circumference of a circle is half the angle subtended at the centre by the same arc. In the diagram shown ∠*AOB* = 2∠*APB*



The arc *AB* **subtends** the angle ∠*AOB* at the centre. The arc also subtends the angle ∠*APB*, called an **angle at the circumference** subtended by the arc *AB*.

Result 5

- Two angles at the circumference subtended by the same arc are equal.



∠APB = ∠AQB

In the diagram, the two angles ∠*APB* and ∠*AQB* are subtended by the same arc *AB*.

Result 6

- The opposite angles of a cyclic quadrilateral are supplementary.

Converse

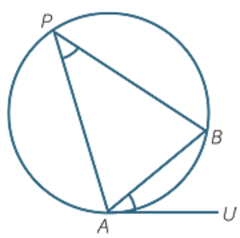
- If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

Result 7

Alternate segment theorem

An angle between a chord and a tangent is equal to any angle in the alternate segment.

In the diagram ∠*BAU* = ∠*APB.*



Unit 2

Trigonometry

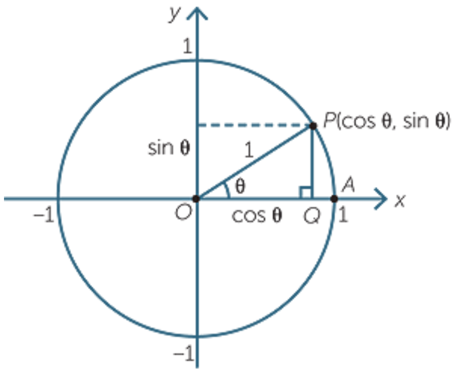
Angle sum and difference identities

$\sin(A + B) = \sin A \cos B + \sin B \cos A$
$\sin(A - B) = \sin A \cos B - \sin B \cos A$
$\cos(A + B) = \cos A \cos B - \sin A \sin B$
$\cos(A - B) = \cos A \cos B + \sin A \sin B$

Cosine and sine functions

Since each angle θ measured anticlockwise from the positive x-axis determines a point P on the unit circle, we will define

- the cosine of θ to be the x-coordinate of the point P
- the sine of θ to be the y-coordinate of the point P
- the tangent of θ is the gradient of the line segment OP



Double angle formula

- $\sin 2A = 2 \sin A \cos A$
- $\cos 2A = \cos^2 A - \sin^2 A$
  $= 2\cos^2 A - 1$
  $= 1 - 2\sin^2 A$
- $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

Products as sums and differences

- $\cos A \cos B = \frac{1}{2}(\cos(A - B) + \cos(A + B))$
- $\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$
- $\sin A \cos B = \frac{1}{2}(\sin(A + B) + \sin(A - B))$
- $\cos A \sin B = \frac{1}{2}(\sin(A + B) - \sin(A - B))$

Pythagorean identities

- $\cos^2 A + \sin^2 A = 1$
- $\tan^2 A + 1 = \sec^2 A$
- $\cot^2 A + 1 = \mathrm{cosec}^2 A$

Reciprocal trigonometric functions

- $\sec A = \dfrac{1}{\cos A}$, $\cos A \neq 0$
- $\operatorname{cosec} A = \dfrac{1}{\sin A}$, $\sin A \neq 0$
- $\cot A = \dfrac{\cos A}{\sin A}$, $\sin A \neq 0$

Matrices

Addition of matrices (See Matrix)
If $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices with the same dimensions and the entries of $\boldsymbol{A}$ are a_{ij} and the entries of $\boldsymbol{B}$ are b_{ij} then the entries of $\boldsymbol{A} + \boldsymbol{B}$ are $a_{ij} + b_{ij}$

For example if $\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 1 & 4 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 5 & 1 \\ 2 & 1 \\ 1 & 6 \end{bmatrix}$ then

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \\ 2 & 10 \end{bmatrix}$$

Determinant of a 2 × 2 matrix (See Matrix)
If $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the determinant of $\boldsymbol{A}$ denoted as $\det \boldsymbol{A} = ad - bc$.
If $\det \boldsymbol{A} \neq 0$,

- the matrix $\boldsymbol{A}$ has an **inverse**.
- the simultaneous linear equations $ax + by = e$ and $cx + dy = f$ have a unique solution.
- The linear transformation of the plane, defined by $\boldsymbol{A}$ maps the unit square

  $O(0, 0)$, $B(0,1)$, $C(1, 1)$, $D(1, 0)$ to a parallelogram $OB'C'D'$ of area $|\det \boldsymbol{A}|$.
- The sign of the determinant determines the orientation of the image of a figure under the transformation defined by the matrix.

Dimension (or size) (See Matrix)
Two matrices are said to have the same **dimensions** (or **size**) **if** they have the same number of rows and columns.

- For example, the matrices

  $\begin{bmatrix} 1 & 8 & 0 \\ 2 & 5 & 7 \end{bmatrix}$ and $\begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$
  have the same dimensions. They are both 2×3 matrices.
- An $m \times n$ matrix has m rows and n columns.

Entries (Elements) of a matrix

The symbol a_{ij} represents the (i, j) entry which occurs in the i^{th} row and the j^{th} column. For example a general 3 × 2 matrix is:

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ and a_{32} is the entry in the third row and the second column.

Leading diagonal

The leading diagonal of a square matrix is the diagonal which runs from the top left corner to the bottom right corner.

Linear transformation defined by a 2 × 2 matrix

The matrix multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

defines a transformation $T(x, y) = (ax + by, cx + dy)$.

Linear transformations in 2 dimensions

A linear transformation in the plane is a mapping of the form

$T(x, y) = (ax + by, cx + dy)$.

A transformation T is linear if and only if

$T(\alpha(x_1, y_1) + \beta(x_2, y_2)) = \alpha T((x_1, y_1)) + \beta T(x_2, y_2))$.

Linear transformations include:

- rotations around the origin
- reflections in lines through the origin
- dilations.

Translations are not linear transformations.

Matrix (matrices)

A **matrix** is a rectangular array of elements or entries displayed in rows and columns.

For example,

$\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 1 & 4 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 1 & 8 & 0 \\ 2 & 5 & 7 \end{bmatrix}$ are both matrices.

Matrix $\boldsymbol{A}$ is said to be a 3 × 2 matrix (three rows and two columns) while $\boldsymbol{B}$ is said to be a 2 × 3 matrix (two rows and three columns).

A **square matrix** has the same number of rows and columns.

A **column matrix** (or vector) has only one column.

A **row matrix** (or vector) has only one row.

Matrix algebra of 2 × 2 matrices

If ***A***, ***B*** and ***C*** are 2 × 2 matrices, ***I*** the 2 × 2 (multiplicative) identity matrix and ***O*** the 2 × 2 zero matrix then:

$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$	(commutative law for addition)
$(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$	(associative law for addition)
$\boldsymbol{A} + \boldsymbol{O} = \boldsymbol{A}$	(additive identity)

$\boldsymbol{A} + (-\boldsymbol{A}) = \boldsymbol{O}$	(additive inverse)
$(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$	(associative law for multiplication)
$\boldsymbol{AI} = \boldsymbol{A} = \boldsymbol{IA}$	(multiplicative identity)
$\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$	(left distributive law)
$(\boldsymbol{B} + \boldsymbol{C})\boldsymbol{A} = \boldsymbol{BA} + \boldsymbol{CA}$	(right distributive law)

Matrix multiplication

Matrix multiplication is the process of multiplying a matrix by another matrix. The product ***AB*** of two matrices ***A*** and ***B*** with **dimensions** $m \times n$ and $p \times q$ is defined if $n = p$. If it is defined, the product ***AB*** is an $m \times q$ matrix and it is computed as shown in the following example.

$$\begin{bmatrix} 1 & 8 & 0 \\ 2 & 5 & 7 \end{bmatrix}\begin{bmatrix} 6 & 10 \\ 11 & 3 \\ 12 & 4 \end{bmatrix} = \begin{bmatrix} 94 & 34 \\ 151 & 63 \end{bmatrix}$$

The entries are computed as shown

$1 \times 6 + 8 \times 11 + 0 \times 12 = 94$

$1 \times 10 + 8 \times 3 + 0 \times 4 = 34$

$2 \times 6 + 5 \times 11 + 7 \times 12 = 151$

$2 \times 10 + 5 \times 3 + 7 \times 4 = 63$

The entry in row i and column j of the product ***AB*** is computed by 'multiplying' row i of ***A*** by column j of **B** as shown.

If $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$ then

$$\boldsymbol{AB} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} \end{bmatrix}$$

(Multiplicative) identity matrix

A **(multiplicative) identity matrix** is a square matrix in which all the elements in the leading diagonal are 1s and the remaining elements are 0s. Identity matrices are designated by the letter ***I***.

For example,

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are both identity matrices.

There is an identity matrix for each order of square matrix. When clarity is needed, the order is written with a subscript: $\boldsymbol{I}_n$

Multiplicative inverse of a square matrix

The inverse of a square matrix ***A*** is written as $\boldsymbol{A}^{-1}$ and has the property that

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$$

Not all square matrices have an inverse. A matrix that has an inverse is said to be **invertible**.

multiplicative inverse of a 2 × 2 matrix

The **inverse** of the matrix $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\boldsymbol{A}^{-1} = \dfrac{1}{\det \boldsymbol{A}}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, when $\det \boldsymbol{A} \neq 0$.

Scalar multiplication (matrices)

Scalar multiplication is the process of multiplying a matrix by a scalar (number).

For example, forming the product:

$$10\begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 20 & 10 \\ 0 & 30 \\ 10 & 40 \end{bmatrix}$$

is an example of the process of scalar multiplication.

In general for the matrix $\boldsymbol{A}$ with entries a_{ij} the entries of $k\boldsymbol{A}$ are ka_{ij}.

Singular matrix

A matrix is singular if det $\boldsymbol{A}$ **=** 0. A singular matrix does not have a multiplicative inverse.

Zero matrix

A zero matrix is a matrix if all of its entries are zero. For example:

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are zero matrices.

There is a zero matrix for each **size** of matrix. When clarity is needed we write $\mathbf{O}_{n\times m}$ for the $n \times m$ zero matrix.

Real and Complex Numbers

Complex numbers

Complex arithmetic

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$

- $z_1 + z_2 = x_1 + x_2 + (y_1 + y_2)\, i$
- $z_1 - z_2 = x_1 - x_2 + (y_1 - y_2)\, i$
- $z_1 \times z_2 = x_1 x_2 - y_1 y_2 + (x_1 y_2 + x_2 y_1) i$
- $z_1 \times (0 + 0i) = 0$ Note: $0 + 0i$ is usually written as 0
- $z_1 \times (1 + 0i) = z_1$ Note: $1 + 0i$ is usually written as 1

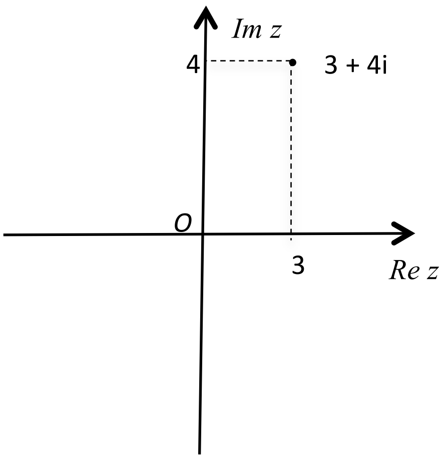
Complex conjugate

For any complex number $z = x + iy$, its **conjugate** is $\overline{z} = x - iy$. The following properties hold

- $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
- $\overline{z_1/z_2} = \overline{z_1} / \overline{z_2}$
- $z\,\overline{z} = |z|^2$
- $z + \overline{z}$ is real

Complex plane (Argand plane)

The **complex plane** is a geometric representation of the complex numbers established by the **real axis** and the orthogonal **imaginary axis**. The complex plane is sometimes called the Argand plane.



Imaginary part of a complex number
A complex number z may be written as $x + yi$, where x and y are real, and then y is the imaginary part of z. It is denoted by $Im\,(z)$.

Integers
The **integers** are the numbers $\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$.

Modulus (Absolute value) of a complex number
If z is a complex number and $z = x + iy$ then the modulus of z is the distance of z from the origin in the Argand plane. The modulus of z denoted by $|z| = \sqrt{x^2 + y^2}$.

Prime numbers
A prime number is a positive integer greater than 1 that has no positive integer factor other 1 and itself. The first few prime numbers are $2, 3, 5, 7, 11, 13, 17, 19, 23, \cdots$.

Principle of mathematical induction
Let there be associated with each positive integer n, a proposition $P(n)$.
If

1. $P(1)$ is true, and
2. for all k, $P(k)$ is true implies $P(k + 1)$ is true,

then $P(n)$ is true for all positive integers n.

Rational numbers
A real number is **rational** if it can be expressed as a quotient of two integers. Otherwise it is called irrational.
Irrational numbers can be approximated as closely as desired by rational numbers, and most electronic calculators use a rational approximation when performing calculations involving an irrational number.

Real numbers
The numbers generally used in mathematics, in scientific work and in everyday life are the **real numbers**. They can be pictured as points on a number line, with the integers evenly spaced along the line, and a real number a to the right of a real number b if $a > b$.
A real number is either rational or irrational. The set of real numbers consists of the set of all rational and irrational numbers.
Every real number has a decimal expansion. Rational numbers are the ones whose decimal expansions are either terminating or eventually recurring.

Real part of a complex number
A complex number z may be written as $x + yi$, where x and y are real, and then x is the real part of z. It is denoted by $Re(z)$.

Whole numbers
A **whole number** is a non–negative integer, that is, one of the numbers $0, 1, 2, 3, \cdots$,

Unit 3

Complex Numbers

Argument (abbreviated arg)
If a complex number is represented by a point P in the complex plane then the argument of z, denoted arg z, is the angle θ that OP makes with the positive real axis O_x, with the angle measured anticlockwise from O_x. The **principal value** of the argument is the one in the interval $(-\pi, \pi]$.

Complex arithmetic (see Complex arithmetic Unit 2)

Complex conjugate (see Complex conjugate Unit 2)

De Moivre's Theorem
For all integers n, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Modulus of a complex number (Modulus of a complex number Unit 2)

Polar form of a complex number
For a complex number z, let $\theta = \arg z$. Then $z = r(\cos\theta + i\sin\theta)$ is the polar form of z.

Root of unity (nth root of unity)
A complex number z such that $z^n = 1$
The n^{th} roots of unity are:
$\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ where $k = 0, 1, 2, \ldots, n-1$.
The points in the complex plane representing roots of unity lie on the unit circle.
The cube roots of unity are
$z_1 = 1$, $z_2 = \frac{1}{2}(-1 + i\sqrt{3})$, $z_3 = \frac{1}{2}(-1 - i\sqrt{3})$. Note $z_3 = \overline{z_2}$ and $z_3 = \frac{1}{z_2}$ and $z_2 z_3 = 1$.

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Functions and sketching graphs

Rational function

A rational function is a function such that $f(x) = \frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials. Usually $g(x)$ and $h(x)$ are chosen so as to have no common factor of degree greater than or equal to 1, and the domain of f is usually taken to be $R\backslash\{x: h(x) = 0\}$.

Vectors in three-dimensions

(See Vectors in Unit 2)

Addition of vectors

In component form if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then
$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}$

Vector equation of a straight line

Let **a** be the position vector of a point on a line l, and **u** any vector with direction along the line. The line consists of all points P whose position vector **p** is given by
$\mathbf{p} = \mathbf{a} + t\mathbf{u}$ for some real number t.
(Given the position vectors of two points on the plane **a** and **b** the equation can be written as
$\mathbf{p} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$ for some real number t.)

Vector equation of a plane

Let **a** be a position vector of a point A in the plane, and **n** a normal vector to the plane . Then the plane consists of all points P whose position vector **p** satisfies
$(\mathbf{p} - \mathbf{a}).\mathbf{n} = \mathbf{0}$. This equation may also be written as $\mathbf{p}.\mathbf{n} = \mathbf{a}.\mathbf{n}$, a constant.
(If the the normal vector **n** is the vector (l, m, n) in ordered triple notation and the scalar product $\mathbf{a}.\mathbf{n} = k$, this gives the Cartesian equation $lx + my + nz = k$ for the plane)

Vector function

In this course a vector function is one that depends on a single real number parameter t, often representing time, producing a vector $\boldsymbol{r}(t)$ as the result. In terms of the standard unit vectors **i**, **j**, **k** of three dimensional space, the vector-valued functions of this specific type are given by expressions such as
$\boldsymbol{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$
where f, g and h are real valued functions giving coordinates.

Scalar product

If $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ then the scalar product **a**.**b** is the real number
$a_1b_1 + a_2b_2 + a_3b_3$.
When expressed in **i**, **j**, **k** notation, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then
$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector product (Cross product)

When expressed in **i**, **j**, **k** notation, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then
$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$
The cross product has the following geometric interpretation. Let **a** and **b** be two non- parallel vectors then $|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram defined by **a** and **b** and
$\mathbf{a} \times \mathbf{b}$ is normal to this parallelogram.
(The cross product of two parallel vectors is the zero vector.)

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Unit 4

Integration and applications of integration

Inverse Trigonometric functions

The inverse sine function, $y = \sin^{-1} x$

If the domain for the sine function is restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ a one to one function is formed and so an inverse function exists.

The inverse of this restricted sine function is denoted by $\sin^{-1}$ and is defined by:

$$\sin^{-1}: [-1, 1] \rightarrow R, \sin^{-1} x = y \text{ where } \sin y = x, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$\sin^{-1}$ is also denoted by arcsin.

The inverse cosine function, $y = \cos^{-1} x$

If the domain of the cosine function is restricted to $[0, \pi]$ a one to one function is formed and so the inverse function exists.

$\cos^{-1} x$, the inverse function of this restricted cosine function, is defined as follows:

$$\cos^{-1}: [-1, 1] \rightarrow R, \cos^{-1} x = y \text{ where } \cos y = x, y \in [0, \pi]$$

$\cos^{-1}$ is also denoted by arccos.

The inverse tangent function, $y = \tan^{-1} x$

If the domain of the tangent function is restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$ a one to one function is formed and so the inverse function exists.

$$\tan^{-1}: R \rightarrow R, \tan^{-1} x = y \text{ where } \tan y = x, y \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

$\tan^{-1}$ is also denoted by arctan.

Rates of change and differential equations

Implicit differentiation

When variables x and y satisfy a single equation, this may define y as a function of x even though there is no explicit formula for y in terms of x. **Implicit differentiation** consists of differentiating each term of the equation as it stands and making use of the chain rule. This can lead to a formula for $\frac{dy}{dx}$. For example,

if $x^2 + xy^3 - 2x + 3y = 0$,

then $2x + x(3y^2)\frac{dy}{dx} + y^3 - 2 + 3\frac{dy}{dx} = 0$,

and so $\frac{dy}{dx} = \frac{2-2x-y^3}{3xy^2+3}$.

Linear momentum

The linear momentum **p** of a particle is the vector quantity **p** = m**v** where m is the mass and **v** is the velocity.

Logistic equation
The logistic equation has applications in a range of fields, including biology, biomathematics, economics, chemistry, mathematical psychology, probability, and statistics.
One form of this differential equation is:

$\frac{dy}{dt} = ay - by^2$ (where $a > 0$ and $b > 0$)

The general solution of this is

$y = \frac{a}{b + Ce^{-at}}$, where C is an arbitrary constant.

Separation of variables

Differential equations of the form $\frac{dy}{dx} = g(x)h(y)$ can be rearranged as long as $h(y) \neq 0$ to obtain

$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$.

Slope field
Slope field (**direction or gradient field**) is a graphical representation of the solutions of a linear first-order differential equation in which the derivative at a given point is represented by a line segment of the corresponding slope

Statistical Inference

Continuous random variable
A random variable X is called continuous if its set of possible values consists of intervals, and the chance that it takes any point value is zero (in symbols, if $P(X = x) = 0$ for every real number x). A random variable is continuous if and only if its cumulative probability distribution function can be expressed as an integral of a function.

Probability density function
The probability density function (pdf) of a continuous random variable is the function that when integrated over an interval gives the probability that the continuous random variable having that pdf, lies in that interval. The **probability density function** is therefore the derivative of the (cumulative probability) distribution function.

Precision
Precision is a measure of how close an estimator is expected to be to the true value of the parameter it purports to estimate.

Independent and identically distributed observations
For independent observations, the value of any one observation has no effect on the chance of values for all the other observations. For identically distributed observations, the chances of the possible values of each observation are governed by the same probability distribution.

Random sample
A random sample is a set of data in which the value of each observation is governed by some chance mechanism that depends on the situation. The most common situation in which the term “random sample” is used refers to a set of independent and identically distributed observations.
Sample mean the arithmetic average of the sample values